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# How tidal forces cause ocean tides in the equilibrium theory

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## Abstract

We analyze why it is erroneous to think that a tidal bulge is formed by pulling the water surface directly up by a local vertical tidal force. In fact, ocean tides are caused by the global effect of the horizontal components of the tidal forces.

## 1. Introduction

Ocean tides are dynamic, but the equilibrium tide theory, which was first proposed by Newton in his great work *Principia*, assumes they are always static. Nevertheless, this idealized theory explains many features of the ocean tides and is still the easiest model for college students. It is suggested on the basis of assuming that the Earth is a perfect sphere with no daily rotation, no landmasses and incompressible, frictionless water covers its surface uniformly (before tides occur).

Regarding the cause of a tidal bulge, one may think intuitively that the ocean surface is pulled directly upwards by a local vertical tidal force (VTF). Some authors have pointed out this is not the case [1, 2] and actually ocean tides are caused primarily by the horizontal components of the tidal forces [1 – 4]. The aim of this paper is, under the framework of the equilibrium theory, to re-examine this matter by presenting arguments at the college level. Analysis on tides can be carried out by the force method or the potential method. We adopt the former since it is more prevalent at the introductory level and more advantageous to understand the mechanism of the cause.

## 2. The Popular Explanation

Without loss of generality, we only consider the lunar tides. The gravitational attraction between the Earth and the Moon is $GM_{moon}M_{earth}/R^2$, where $G$ is the universal gravitational constant, $R$ is the centre-to-centre Moon-Earth separation, and $M_{moon}$ and $M_{earth}$ are the masses of the Moon and the Earth, respectively. This force acts mutually, so the Moon and the Earth rotate together about their common centre of mass (CM). It is noteworthy that when the centre of the Earth revolves about CM, any diameter of the Earth will keep its direction relative to distant stars unchanged [1, 3]. The above mentioned force produces an acceleration of the Earth, $a_{earth} = GM_{moon}/R^2$. Consider that a mass $m$ is put on the Earth's surface closest to the Moon (the sublunar point, see figure 2). The Moon also attracts $m$ and makes it accelerate, $a_{mass} = GM_{moon}/(R\text{-}R_E)^2$, where $R_E$ is the radius of the Earth and hence $R\text{-} R_E$ is the distance between $m$ and the Moon's centre. Both $m$ and the Earth accelerate in the same direction, so as one "sits" on the Earth to see, $m$ accelerates at $a_{mass} - a_{earth} = GM_{moon}[1/(R\text{-}R_E)^2\text{-} 1/R^2] \sim 10^{-6}$ ms$^{-2}$ (towards the Moon). When $m$ is relocated to the diametrically opposite side of the Earth, i.e. the furthest point from the Moon (the antipodal point), its acceleration relative to the Earth becomes $a_{mass} - a_{earth} = GM_{moon}[1/(R+R_E)^2\text{-} 1/R^2] \sim -10^{-6}$ ms$^{-2}$ (away from the Moon). In other words, no matter $m$ is put at the sublunar or antipodal point, its acceleration seen from the Earth is always directly outward to enable it to leave the Earth (actually $m$ does not leave because of its weight, see the last paragraph of this section). In this way, the ocean waters are vertically lifted up and two high tides of approximately the same size are produced at these two places. Explanations like this are commonly found in texts and on the Internet. Because $a = F/m$, the acceleration $10^{-6}$ ms$^{-2}$ can be interpreted as the VTF per unit mass, which is denoted in this paper as $\gamma$.

In mechanics, we know that in an accelerating frame of reference (FR), Newton's second law is applicable only when each observed object is thought to be acted by an inertial (or fictitious) force. From the above, we apply $a = F/m$ in an accelerating FR (the Earth accelerates at $a_{earth}$, it is the FR) to deduce a tidal force. So this tidal force must contain an inertial force. Nevertheless, ocean water "feels" this inertial force exactly as a real force, just like a passenger in a car following a circular track (centripetal acceleration) feels very truly that he or she is pushed outwards (by an inertial force of exactly the same type). Simply speaking, when we "sit" on the Earth to see, tidal forces can be completely treated as normal real forces.

So far we have neglected a major force on $m$, the gravitational pull from the Earth. Weight does not contribute to the tidal force, but it plays an essential role in establishing equilibrium. As compared with the Earth's gravity ($g$ = 9.81 ms$^{-2}$), $\gamma$ is only about $10^{-7}$ of its value. When water is piled up, it must be simultaneously counteracted by its own weight. The issue is both the VTF and weight are proportional to mass $m$ and, therefore, no matter how high a tidal bulge is, their ratio is still $10^{-7}$ in 1. In view of this, it is difficult to convince one to believe that such a vertical force can produce the ocean tide that is observed in reality.

## 3. Vertical Bulge Caused by Horizontal Force

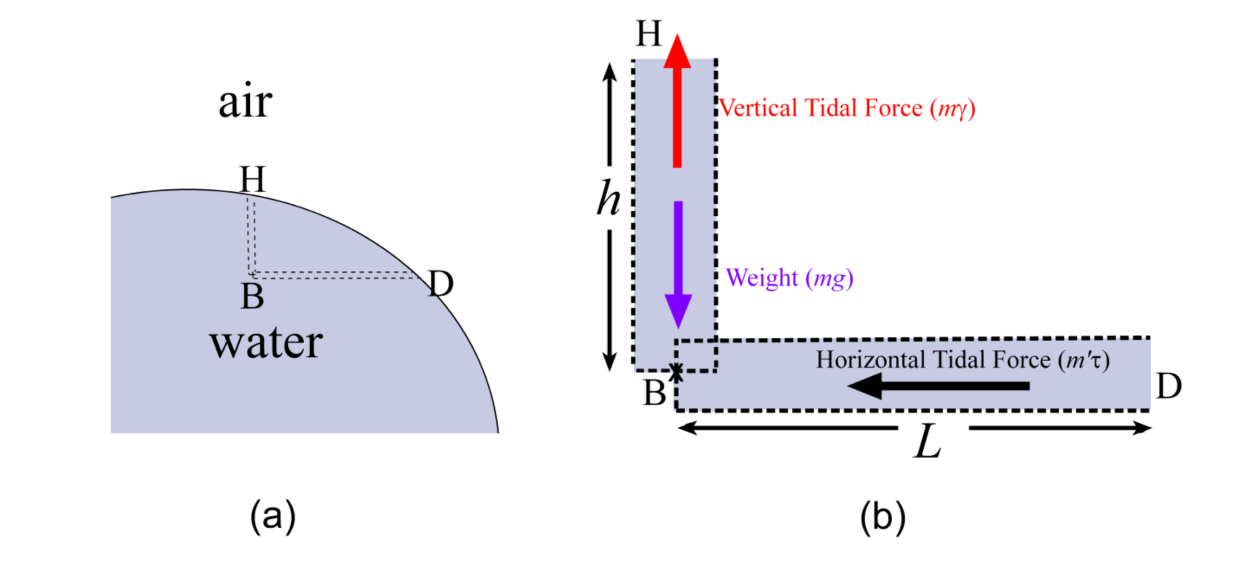


Figure 1. (a) Part of a tidal bulge. (b) Under the condition of static equilibrium, the forces involved must satisfy a condition.

Figure 1(a) shows part of a static tidal bulge. H and D are, respectively, a higher and lower point on the surface, B is the point directly beneath H such that HB is perpendicular to BD. Consider two imaginary no-physical-boundaries tubes of cross-sectional area $A$; one is vertical of length $h$ connecting H and B and one is horizontal of length $L$ connecting B and D, as shown in figure 1(b). The masses of water contained in the vertical and horizontal tubes are $m$ and $m'$ respectively. The water in the vertical tube is influenced by two forces $mg$ and $m\gamma$, or equivalently, a downward net force $m(g - \gamma) = \rho Ah(g\text{-}\gamma)$, where $\rho$ is the density of water. This part of water is static, so the excess water pressure set up at its bottom (point B) is $\rho h(g\text{-}\gamma)$, which will inevitably in turn push the water in the horizontal tube to the right. The water in the horizontal tube is static too, so an additional horizontal force must be there to act on the water from D to B to achieve a balance. Yes, this is the horizontal component of the tidal force. If we denote this *horizontal tidal force* (HTF) per unit mass as τ, then the excess pressure set up at B by this horizontal force is $m'\tau/A = \rho L\tau$. Provided that there is an overall balance, the condition $\rho h(g\text{-}\gamma)= \rho L\tau$ must be satisfied with different pairs of $h$ and $L$. Express the condition as $h = L\tau/(g - \gamma)= ( L\tau/g)/(1 - \gamma/g)$. The term $\gamma/g$ in the denominator can be safely discarded because $\gamma/g \sim 10^{-7}$, therefore

$$h=\frac{L\tau}{g}. \qquad (1)$$

Note that equation (1) is independent of $A$, so $L$ is taken as the length of the line segment parallel to $\tau$. This line segment is not necessary straight and can be arbitrarily long since the water is assumed to be static and ideal. If $\tau$ is not constant over $L$, we can either replacing the numerator by an integral or regard $\tau$ as the spatial average. Equation (1), which can also be derived by the work-energy theorem, is the core of our argument. It simply implies that *a tidal elevation (h) is produced because of the existence of the horizontal tidal force ($\tau$) over a horizontal distance (L) while the vertical tidal force ($\gamma$) is legitimately ignored.* In addition, $\gamma$ and $\tau$ have the same order of magnitude since they are components of the same tidal force. A considerable tidal bulge is formed only when $L$ is large enough because $\tau << g$.

## 4. Horizontal Tidal Force (HTF)

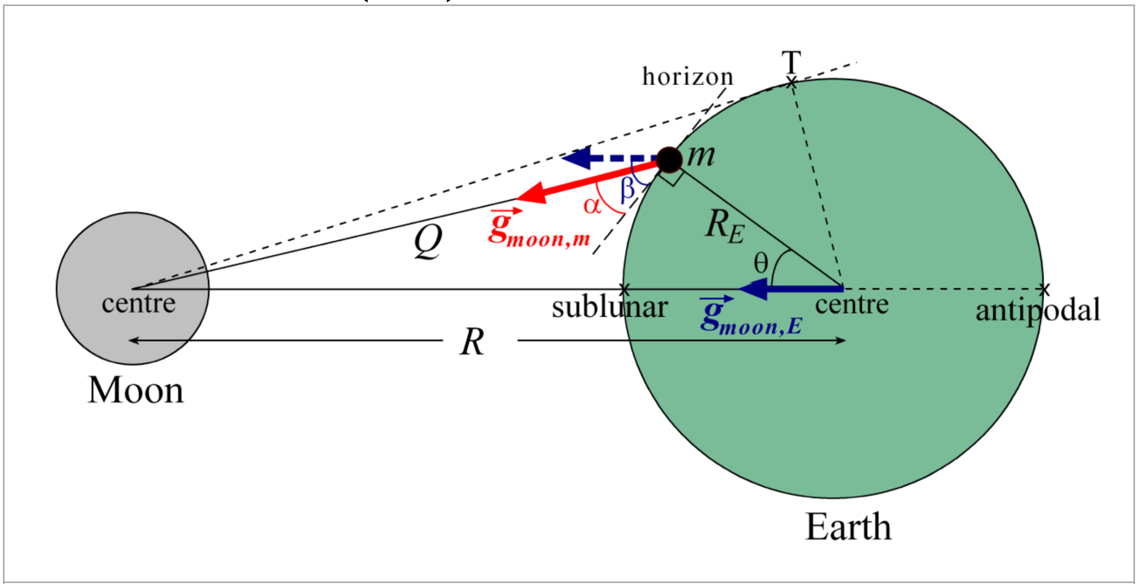


Figure 2. The two non-collinear forces of magnitudes $g_{moon,\,m} = GM_{moon}/Q^2$ and $g_{moon,\,E} = GM_{moon}/R^2$ are the lunar gravitational attraction per mass on $m$ and on the Earth respectively. The vector subtraction of the latter from the former gives the tidal force. The sizes of the objects and the distances are not drawn in scale.

Our analysis relies heavily on how the HTFs vary and direct throughout the Earth's surface. Hence, we have to figure out the tidal force at any place on the Earth's surface. Figure 2 shows a *unit mass m*, which is placed on the Earth's surface and makes an angle $\theta$ with the Moon-Earth joining line.

A basic notion of the ocean tides due to, e.g. the Moon, is that this heavenly object sets up a non-uniform gravitational field (force per mass) over the whole Earth, as evidenced by the tidal forces per mass found in the section "The Popular Explanation", i.e. $GM_{moon}/(R \pm R_E)^2$-$GM_{moon}/R^2$. Based on the same notion, we can derive the general expression by just modifying the particular ones. The first term, $GM_{moon}/(R \pm R_E)^2$, referring to the lunar gravitational attraction per mass on $m$, should now be changed to $GM_{moon}/Q^2$, where $Q$ is the distance between $m$ at its present position and the Moon's centre. The second term, $GM_{moon}/R^2$, referring to the lunar gravitational attraction per mass on the Earth, should be the same since it is irrelevant to $m$. Henceforth, we denote these two terms as $g_{moon,\,m}$ and $g_{moon,\,E}$ respectively. Mathematically, they are no longer subtracted algebraically because they are in general non-collinear. As shown in figure 2, $\vec{g}_{moon,\, m}$ directs towards the Moon's centre from $m$ while $\vec{g}_{moon,\, E}$ directs towards the Moon's centre from the Earth's centre. Therefore, the tidal force is obtained by subtracting $\vec{g}_{moon,\, E}$ from $\vec{g}_{moon,\, m}$ *vectorially*. Here, we employ the components method to carry out the vector subtraction, in order to see clearly how the horizontal component of the tidal force directs.

Displace the vector $\vec{g}_{moon,\, E}$ to $m$ (the dotted line blue arrow in figure 2) for an easier comparison between it and $\vec{g}_{moon,\, m}$. With the unit mass placed at that particular position shown in figure 2, we see that $Q < R$ and therefore $g_{moon,\, m} > g_{moon,\, E}$ (the inverse square law). The horizontal components (relative to the local horizon) of these two forces, in terms of the angles $\alpha$ and $\beta$ defined in figure 2, are $g_{moon,\, m} \cos\alpha$ and $g_{moon,\, E} \cos\beta$, respectively. In addition, $\alpha < \beta$ (see figure 2) and hence $\cos\alpha > \cos\beta$. By combining the two inequalities $g_{moon,\, m} > g_{moon,\, E}$ and $\cos\alpha > \cos\beta$, we get the HTF per unit mass, $\tau = g_{moon,\, m} \cos\alpha - g_{moon,\, E} \cos\beta > 0$. Therefore, $\tau$ at that point is pointing anticlockwise towards the Moon-Earth line. Such a conclusion can be made in the upper hemisphere within the range from the sublunar point ($\theta = 0^0$) to the upper point $T$ at where $Q = R$. Moreover, the Moon is far away from the Earth as compared with the Earth's radius ($R \approx 60R_E$), so the point $T$ is practically at the top of the Earth ($\theta$ for $T = 89.5^0$).

A drawing similar to figure 2, but with the unit mass shifted to the rear side of the upper hemisphere, makes evident $Q > R$ , $\alpha > \beta$ and clockwise HTFs are therefore resulted in this region. Because of the up-down symmetry, the HTFs in the lower hemisphere are the mirror image of the upper's. Finally, we obtain a global picture of the HTFs, which is depicted in figure 3 together with the oval shape of the tidal ocean. Furthermore, a 3D illustration can be created by revolving figure 3 about its horizontal axis of symmetry.

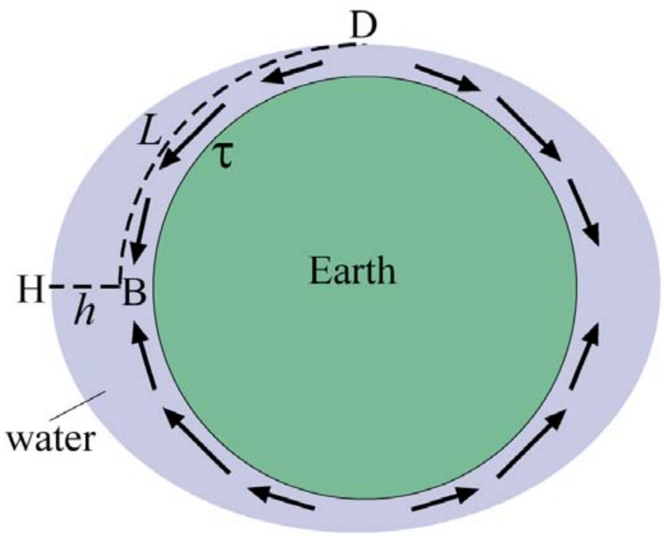


Figure 3. Ocean tides are produced by the horizontal components of the tidal forces (τ). The ocean depth is much exaggerated in the figure.

By applying equation (1) to the dotted path HBD in figure 3, we can perceive the reason for the surface elevation at H. From the low tide D to the high tide H along arc DB, the HTFs ($\tau$) always act in the same direction. In this way, the ocean water is compressed along this angular direction and so the underwater pressure builds up gradually and continually over a large-scale distance, reaching its maximum at B and hence water is bulged high up to form the high tide there. Simply speaking, *ocean tides are the manifestations of the global underwater pressure change caused by the HTFs*. Note that at a high or low tide, the HTF is zero whereas the pressure change is the greatest.

Before an exact calculation, we use equation (1) to estimate $h$, which is the height difference between the high and low tides. As mentioned above, $\gamma$ and $\tau$ are in the same order of magnitude. We have already calculated $\gamma \sim 10^{-6}$ ms$^{-2}$ at the two high tides. In equation (1), we simply take $\tau \sim 10^{-6}$ ms$^{-2}$, $L$ = length of arc DB = one fourth of the Earth's circumference = Earth's radius x ($\pi$/2) ~ $10^7$ m. Thus, we get $h = L\tau/g \sim (10^7)(10^{-6})/9.8 \sim 1$ m. This estimation is rough but instructive. The most determinant factor that makes $h$ discernible is the "one fourth of the Earth's circumference", the distance $\tau$ can act without changing its direction.

## 5. Expressions for VTF and HTF

The expressions for the vertical and horizontal components of the tidal force are derived in this section, making the theory complete and convenient to those readers who are interested to do calculations and analysis of their own.

As terms defined in figure 2, the HTF per unit mass, $\tau = g_{moon,\,m} \cos\alpha - g_{moon,\,E} \cos\beta$ and the VTF per unit mass, $\gamma = g_{moon,\,m} \sin\alpha - g_{moon,\,E} \sin\beta$, where $g_{moon,\,m} = GM_{moon}/Q^2$ and $g_{moon,\,E} = GM_{moon}/R^2$. In carrying out the calculations, we need the following relationships: $\cos\alpha = R\sin\theta/Q$, $\sin\alpha = (R\cos\theta - R_E)/Q$, $\beta = \pi/2 - \theta$ and $Q^2 = R^2 - 2RR_E\cos\theta + R_E^2$ (the law of cosines). Keeping terms only up to the first order of ($R_E/R$) in the expansions of the expressions, we finally obtain

DB. Since D and B are equidistant from the Earth's centre, $\tau$ is parallel to arc DB. In equation (1), $g = GM_{earth}/R_E^2$ and the numerator $L\tau$ is replaced by a definite integral of equation (2) with respect to distance over the arc from D to B. At last, we obtain an expression for $h$, the height difference between the high and low tides,

$$h = \frac{3}{2}\frac{M_{moon}}{M_{earth}}\frac{R_E^4}{R^3} \quad . \tag{4}$$

Equations (2) – (4) agree with the results in the literature [2, 3, 5 – 7]. As we put values into equation (4), we get $h$ ~ 0.5m, of which the order of magnitude has been estimated in the previous section.

## 6. Newton's Wells

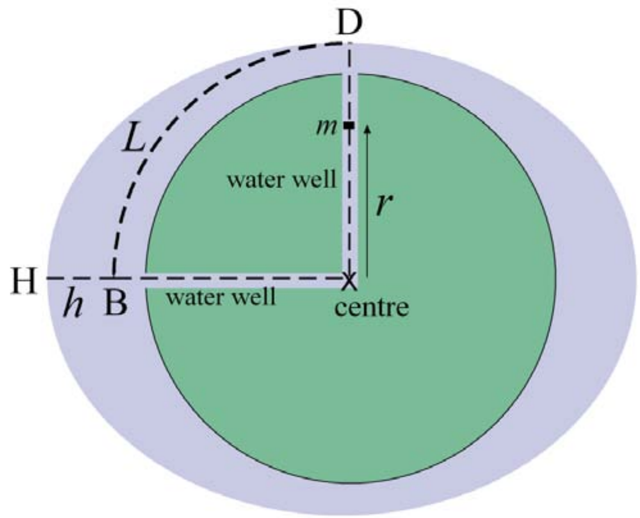


Figure 4. Newton devised an ingenious method to derive the tidal height, $h$. In his method, two imaginary water wells are drilled from the ocean surface to the Earth's centre. The ocean depth is much exaggerated in the figure.

Nowadays, texts are still apt to use a method devised by Newton to derive the tidal height. In his method, two perpendicular water wells are imagined to be drilled, one runs from a low tide to the Earth's centre and joins the other which runs from the Earth's centre to a high tide [5], as shown in figure 4. Then a mass of water $m$ is transferred from the low tide to the high tide via the two wells, the tidal height ($h$) is obtained by equating the change in gravitational potential energy ($mgh$) to the net work done by the tidal force, of which only the vertical component is involved in this case. The tidal forces on $m$ along the two wells are obtained from our equation (3) by substituting $\theta = 90^0$ and $0^0$, respectively, and changing $R_E$ to a variable, named as $r$ in figure 4. Apparently, this approach contradicts with our claim of the unimportance of the VTF.

There is no doubt that only the VTF does work because $m$ is moved along the two deep-to-the-Earth's centre water wells. But in reality these two wells do not exist, they are imaginary and only for the good of deriving the tidal height. Our claim applies to the real case: ocean water of negligible depth, as compared with the size of the Earth, is restricted only on the Earth's surface.

Newton's method derives the correct result because the tidal force, which origins from the gravitational attraction, is conservative and so the work done by it is path-independent. In figure 4, either the angular path $L$ or the path through the two wells is taken, the net work done by the tidal force from D to B (or the water pressure difference between D and B) must be the same. Remember that only the path $L$ seems to be practically possible.

## 7. Discussion and Conclusion

The horizontal components of the tidal forces are responsible for the ocean tides, while their vertical counterparts play an insignificant role. There are two reasons for this. First, only the latter are counteracted disproportionately by the much larger Earth's gravity. Secondly, the former benefit from the spatial extensions of the shell-like ocean water, whose radial span (~ ocean depth) is very limited as compared with its angular span (~ size of Earth). Although the horizontal components are as feeble as the vertical, they are compensated with the "one fourth of the Earth's circumference".

One may argue if the underwater static pressure really rises continually over an exceedingly long distance ~ $10^7$m to produce a tidal bulge ~ 1 m. Of course, this is the scenario resulted from and seen as a necessary part of the self-contained equilibrium tide theory. Real ocean tides are described more satisfactorily by the dynamic theory. Nevertheless, the reasons listed in the first paragraph of this section always prevail, so it is very model-independent to say the horizontal components of the tidal forces cause the ocean tides. In the dynamic model, these forces drive ocean water move to set up tidal currents to transport water from the low tides to the high tides [1 – 4].

In short, *a tidal force is too feeble to cause a local ocean tide, which is the outcome of the global action of the horizontal components of the tidal forces.*

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