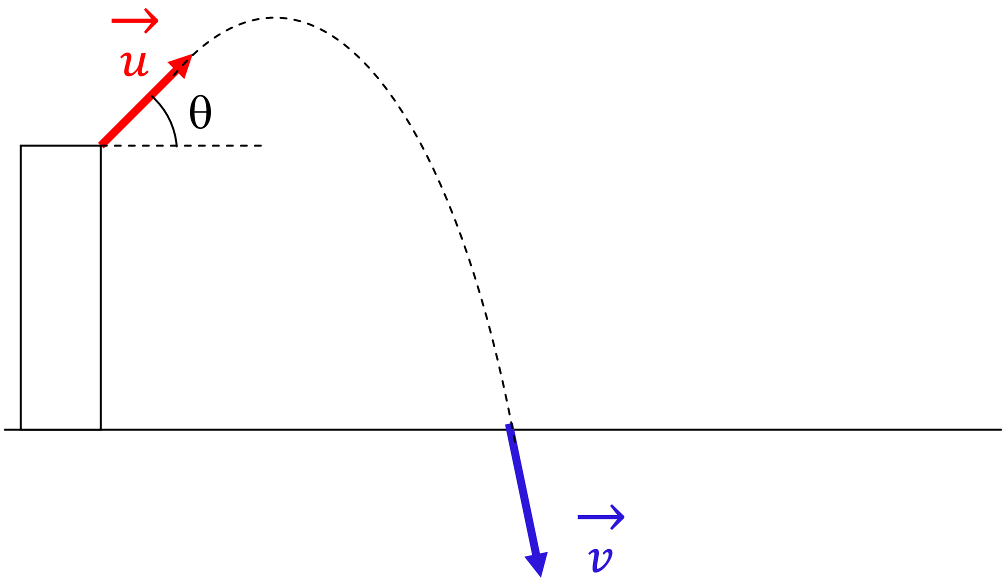
A Non-Calculus Derivation of the Maximum Range

Let the launching velocity be $\vec{u}$, the launching direction measured from the horizontal be θ, the landing velocity be $\vec{v}$ and the time of flight be **t.**



In midair, object is acted on by gravity only, so

$$\vec{v} = \vec{u} + \vec{g}t,$$

which can be represented by the following vector diagram

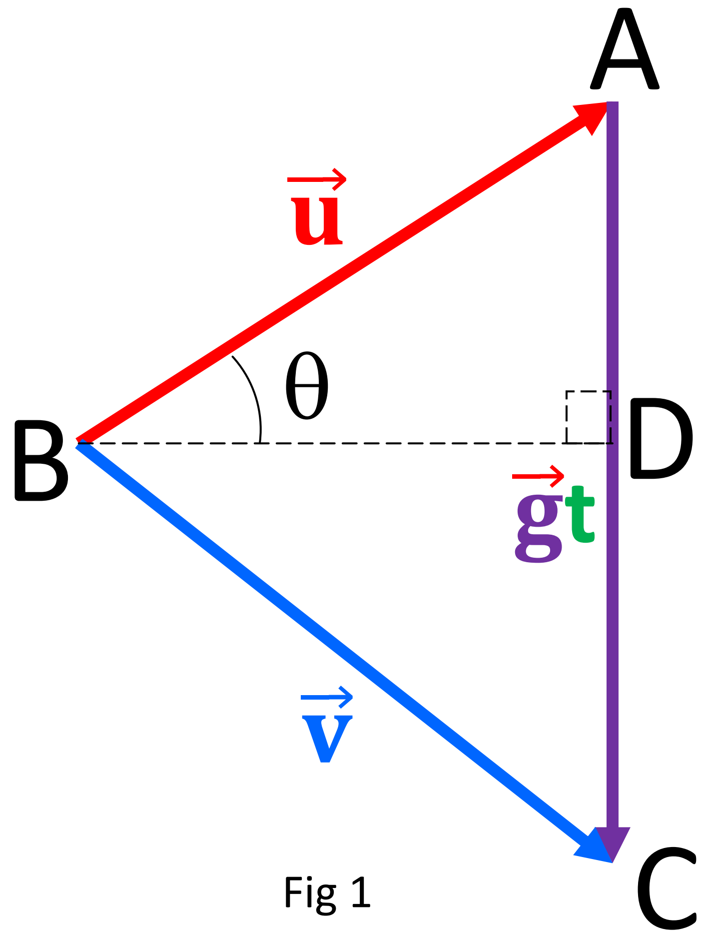


Fig 1

Area of Δ ABC

$$= \frac{1}{2}(AC)(BD)$$

$$= \frac{1}{2}(u\cos\theta)(gt) \qquad \ldots (1)$$

The term $u\cos\theta$ in eq. (1) is the horizontal component of $\vec{u}$, denoted as u_x

Therefore, eq (1) can be rewritten as

Area of $\triangle$ABC = $\frac{1}{2}g(u\cos\theta)(t)$ = $\frac{1}{2}g(u_x t)$ = $\frac{1}{2}gR$,

where $R = u_x t$ is the range.

In other words,

$$\text{Range} = \frac{2 \times \text{area of } \triangle \text{ABC}}{g}$$

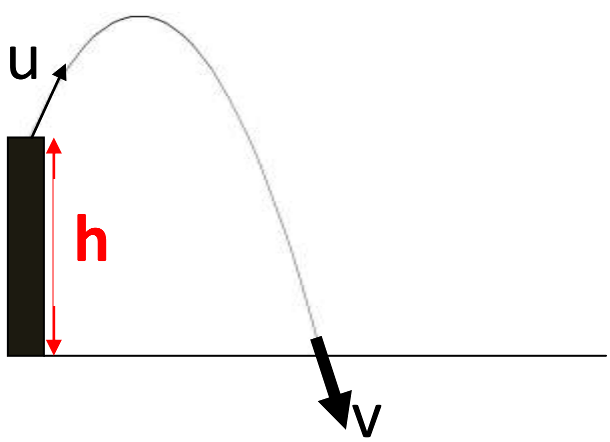
- Hence, R is maximum when △ABC has the largest area.
- At a fixed magnitude of the launching speed (u), the magnitude of the landing speed (v) is also fixed (by the principle of conservation of energy).
- In Fig. 1, the lengths of sides AB and BC are fixed. The area of △ABC is the largest only when ∠ABC is a right angle.

Maximum range is achieved when the landing velocity is perpendicular to the launching velocity

- If $\angle ABC = 90^0$，$\angle ACB = \theta$，$\tan\theta = \frac{u}{v}$
- According to energy conservation,

$$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mgh$$

$$v = \sqrt{u^2 + 2gh}$$.



Under the conditions for maximum range, fig (1) becomes

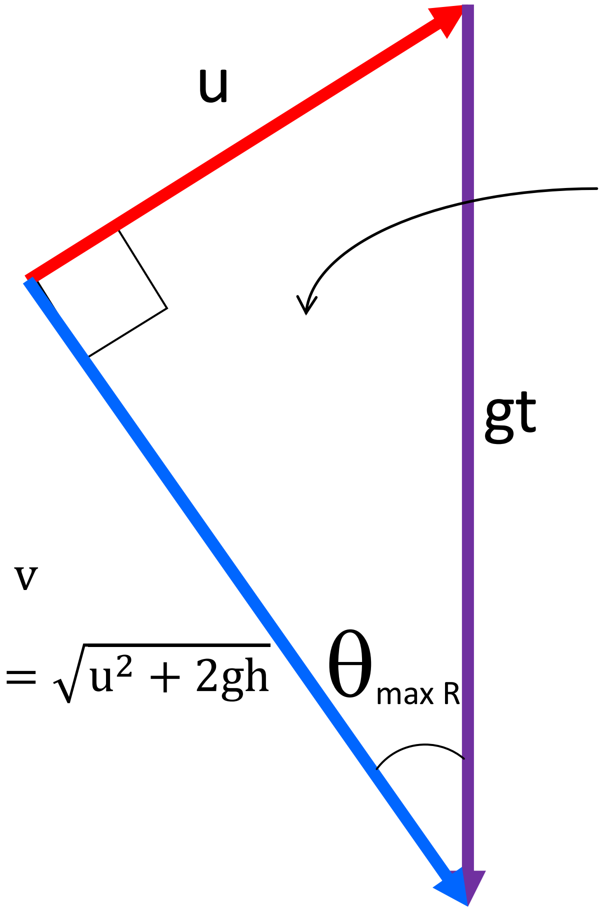


Fig. 2

Fig. 2 tells the whole story. Obviously,

1. The range is maximum when the launching angle is $\theta = \tan^{-1}(\frac{u}{v})$, $\therefore$

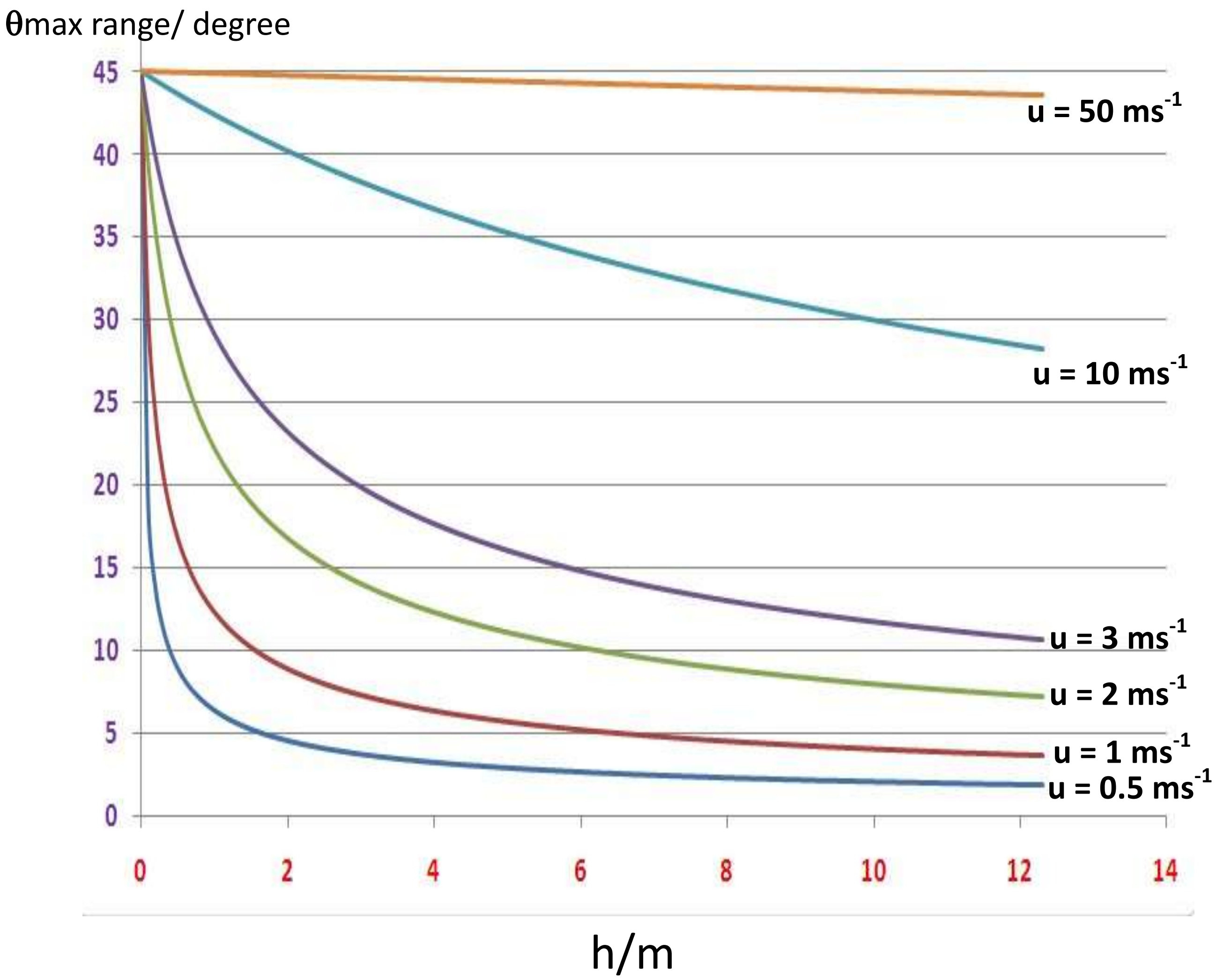
$$\boxed{\theta_{\text{max range}} = \tan^{-1}\left(\frac{u}{\sqrt{u^2+2gh}}\right)}$$

2. The maximum range is $\frac{2\times\text{area of}\triangle\text{ABC}}{g} = \frac{uv}{g}$

$$\boxed{R_{max} = \frac{1}{g}uv = \frac{1}{g}u\sqrt{u^2 + 2gh}}$$

3. The time of flight then is $t = \frac{\sqrt{u^2+v^2}}{g}$

$$t = \frac{\sqrt{2(u^2+gh)}}{g}$$



- $\theta_{max\ range}$ is less than 45^0 unless h = 0.
- For large launching speed ($u^2 >> 2gh$), $\theta_{max\ range} \lesssim 45^0$
- For small launching speed ($u^2 << 2gh$), $\theta_{max\ range} << 45^0$

  For example, h = 1.2 m, u = $4ms^{-1}$, $\theta_{max\ range} = 32^0$
- Since $R_{max} = \frac{1}{g}u\sqrt{u^2 + 2gh}$.

  The higher the firing platform (h) is, the larger is the max range.

  For example,
  - u = 4 ms^{-1}

    h = 0, $\theta_{max\ range} = 45^0$, R_{max} = 1.6m
  - u = 4 ms^{-1}

    h = 1.2m, $\theta_{max\ range} = 32^0$, R_{max} = 2.5m
  - u = 4 ms^{-1}

    h = 1.6m, $\theta_{max\ range} = 30^0$, R_{max} = 2.8m

Two Required Qualities of Shot Put Athletes

1. Muscular (can give a large u)
2. Tall (large h)

Reference:

W.M.Young, Am. J. Phys. 53, 1(1985)